ISRAEL JOURNAL OF MATHEMATICS **135** (2003), 251–268

TIGHT SUBGROUPS IN TORSION-FREE ABELIAN GROUPS

BY

MOHAMED A. OULD-BEDDI

Faculté des Sciences et Techniques, Université de Nouakchott
B. P. 5026, Nouakchott, Mauritania
e-mail: beddima@univ-nkc.mr

AND

LUTZ STRÜNGMANN*

Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
e-mail: lutz.struengmann@uni-essen.de

ABSTRACT

Let X be a torsion-free abelian group. We study the class of all completely decomposable subgroups of X which are maximal with respect to inclusion. These groups are called tight subgroups of X and we state sufficient conditions on a subgroup to be tight. In particular we consider tight subgroups of bounded completely decomposable groups. For those we show that every regulating subgroup is tight and we characterize the tight subgroups of finite index in almost completely decomposable groups.

1. Introduction

A **completely decomposable** group is a direct sum of groups isomorphic to subgroups of the additive group of rational numbers. An **almost completely decomposable group** X is a finite essential (abelian) extension of a completely decomposable group A of finite rank. In [BMO00] Benabdallah, Mader and the first author studied **tight subgroups** of almost completely decomposable groups, i.e., completely decomposable subgroups which are maximal with respect

* The second author was supported by a MINERVA fellowship.
Received November 29, 2001

to inclusion. It turned out that **regulating subgroups** are always tight in an almost completely decomposable group. The regulating subgroups can be defined as the completely decomposable subgroups of least index and E. L. Lady [Lad74] initiated a systematic theory of almost completely decomposable groups based on regulating subgroups. However, there are tight subgroups in almost completely decomposable groups which are not regulating but which have finite index in the overgroup (see [BMO00, Example 2.13]).

In [BMO00] the authors claimed that all tight subgroups of an almost completely decomposable group X have finite index in X but gave an incorrect proof. In fact, the result is not true and we give an example of a tight subgroup in an almost completely decomposable group which does not have finite index (see Example 4.1). Moreover, we characterize those tight subgroups A of almost completely decomposable groups X which have finite index as the sharply tight subgroups, i.e., tight subgroups such that also $A^\sharp(\tau)$ is tight in $X^\sharp(\tau)$ for all types τ (see Corollary 4.3). As a consequence let us remark that all results in [BMO00] remain valid if one replaces tight subgroups by sharply tight subgroups or equivalently by tight subgroups of finite index.

The present paper deals with the class TSgps(X) of all completely decomposable subgroups in an arbitrary torsion-free abelian group X that are maximal with respect to inclusion. These will be called **tight subgroups** of X adopting terminology used by Benabdallah, Mader and the first author. A torsion-free group need not contain tight subgroups as Example 2.2 shows but Theorem 2.9 provides a sufficient condition on a completely decomposable subgroup to be tight. However, our assumptions are not necessary. Nevertheless, if we strengthen the tightness condition on a completely decomposable subgroup A of X to strong tightness, i.e., if we require that for all types τ, $A(\tau)$ is tight in $X(\tau)$, then we obtain a complete characterization of the strongly tight subgroups (see Corollary 2.15).

In section 3 we consider the class of **bounded completely decomposable groups**, i.e., torsion-free groups X which contain a completely decomposable group A with bounded quotient X/A. This class of groups is a natural extension of the class of almost completely decomposable groups and was first studied by Elter in [El96] and later by Mader and the second author in [MS00] (see also [Du01]). Bounded completely decomposable groups possess Butler decompositions and completely decomposable regulating subgroups which are defined as the sum of the Butler complements. We show that regulating subgroups are examples of (even sharply) tight subgroups (see Proposition 3.2) and prove that tight subgroups with elementary quotient are necessarily regulating (see Corollary 3.4).

All groups under consideration are abelian. The purification of a subgroup H in a torsion-free group G is denoted by H_*^G for emphasis. We take it for granted that the reader is familiar with the usual type subgroups $G(\tau)$, $G^*(\tau)$, and $G^\sharp(\tau) = G^*(\tau)_*$. If A is a completely decomposable group, then $A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho$ is assumed to be a decomposition of A into ρ-homogeneous components $A_\rho \neq 0$, so that $\mathrm{T}_{\mathrm{cr}}(A)$ is the **critical typeset** of A. A typeset $\mathrm{Tst}(X)$ of a torsion-free group X satisfies the **maximum condition** if every non-empty subset of $\mathrm{Tst}(X)$ has maximal elements, which is equivalent to saying that every ascending chain in $\mathrm{Tst}(X)$ is finite. Moreover, we will identify rational groups with their types, e.g., $\mathbb{Z} = \mathrm{tp}(\mathbb{Z})$. For background on torsion-free abelian groups and almost or bounded completely decomposable groups see [Arn82], [Mad95], [Mad99], and [MS00]. Maps are written on the right.

2. Tight subgroups of torsion-free groups

In [BMO00] the authors defined tight subgroups of almost completely decomposable groups and studied their properties. Here we extend their definition to arbitrary torsion-free (abelian) groups.

Definition 2.1: Let X be a torsion-free group and A a completely decomposable subgroup of X. Then A is called **tight** in X if it is maximal among the completely decomposable subgroups of X with respect to inclusion.

Note that for a tight subgroup A of X we necessarily have that X/A is torsion. However, tight subgroups need not exist in general as the following example shows.

Example 2.2: Let X be a homogeneous group which is not completely decomposable. Then X contains no tight subgroups.

Proof: Assume that A is a tight subgroup of X. Then the quotient X/A is torsion and A is homogeneous completely decomposable and pure in X by Theorem 2.9 (1) below. Hence $X = A$ — a contradiction. ∎

The following theorem gives some sufficient conditions for tightness in arbitrary torsion-free groups. This criterion is useful in establishing tightness. Some preliminary results are needed.

LEMMA 2.3: *Let B be a torsion-free group and A a homogeneous completely decomposable subgroup of B with the property that every rank-one summand of A is pure in B and B/A is torsion. Then $A = B$.*

Proof: Let $x \in B$; then $nx \in A$ for some integer n since B/A is torsion. By [Fuc73, Theorem 86.8] it follows that $\langle nx \rangle_*^A$ is a rank one summand of A since A is homogeneous. Thus $\langle nx \rangle_*^A = \langle nx \rangle_*^B$, and hence $x \in A$. ∎

For the next proposition recall that a subgroup A of a torsion-free group B is called **regular** in B if for all types τ the equality $A(\tau) = B(\tau) \cap A$ holds. This is equivalent to saying that every element of A has the same type in A as in X (see [Arn81]).

PROPOSITION 2.4: *Let B be a completely decomposable group and A a completely decomposable regular subgroup of B such that B/A is torsion, and assume that* $\mathrm{Tst}(A)$ *satisfies the maximum condition. If every rank-one summand of A is pure in B, then $A = B$.*

Proof: By induction on $\mathrm{Tst}(A)$ we show that for every $\tau \in \mathrm{Tst}(A)$ we have $B(\tau) = A(\tau)$. This is clearly enough to prove $A = B$, because if $x \in B$, then $nx \in A(\tau)$ for some $\tau \in \mathrm{Tst}(A)$. Thus $nx \in B(\tau)$ and hence also $x \in B(\tau) = A(\tau)$. Now, let τ be maximal in $\mathrm{Tst}(A)$. Then $A(\tau)$ is τ-homogeneous completely decomposable and $B(\tau)/A(\tau)$ is torsion by regularity. Thus by Lemma 2.3 we have $B(\tau) = A(\tau)$. Now, assume that $\tau \in \mathrm{Tst}(A)$ and for every $\tau < \sigma \in \mathrm{Tst}(A)$ we have $A(\sigma) = B(\sigma)$. Write $B(\tau) = B_\tau \oplus B^\sharp(\tau)$ with B_τ τ-homogeneous completely decomposable. Then we obtain

$$B^\sharp(\tau) = B^*(\tau) = \sum_{\sigma > \tau} B(\sigma) = \sum_{\sigma > \tau} A(\sigma) = A^\sharp(\tau).$$

Write $A(\tau) = A_\tau \oplus A^\sharp(\tau)$ with A_τ τ-homogeneous completely decomposable. Let $b \in A_\tau$. Then $b = a_b + a_b^\sharp \in B_\tau \oplus B^\sharp(\tau) = B_\tau \oplus A^\sharp(\tau)$ with $a_b \in B_\tau$ and $a_b^\sharp \in A^\sharp(\tau)$. Let $\phi\colon A_\tau \longrightarrow A^\sharp(\tau)$ be the homomorphism with $\phi(b) = a_b^\sharp$. It follows easily that $A(\tau) = A_\tau(1-\phi) \oplus A^\sharp(\tau)$. Moreover, if $b' \in A_\tau(1-\phi)$ then $b' = b - b\phi = b - a_b^\sharp = a_b \in B_\tau$. Thus $A'_\tau \subseteq B_\tau \cap A$, where $A'_\tau = A_\tau(1-\phi)$. Furthermore, the regularity implies that $A'_\tau = B_\tau \cap A$. Hence

$$\frac{B_\tau + A}{A} \cong \frac{B_\tau}{B_\tau \cap A} = \frac{B_\tau}{A'_\tau}$$

is torsion and therefore $B_\tau = A'_\tau$ by Lemma 2.3. Finally, it follows that $B(\tau) = B_\tau \oplus B^\sharp(\tau) = A'_\tau \oplus A^\sharp(\tau) = A(\tau)$. ∎

The next example shows that we cannot remove the assumption on the quotient B/A to be torsion in Proposition 2.4; it also proves that in [BMO00, Lemma 2.6] one has to assume this condition.

EXAMPLE 2.5: *Let* $B = \mathbb{Z}[p^{-1}]x \oplus \mathbb{Z}[q^{-1}]y$ *where* p *and* q *are distinct primes. Moreover, let* $A = \mathbb{Z}(x+y)$. *Then* A *is pure in* B *and hence every rank-one summand of* A *is pure in* B *but* A *is not a direct summand of* B.

Proof: It is clear that A is pure in B and hence every rank-one summand of A is pure in B. Since the type of the integers is not critical in B, A cannot be a direct summand of B. ∎

The following example shows that we cannot drop the assumption of regularity in Proposition 2.4 even in the finite rank case.

EXAMPLE 2.6: *Let* l, p, r, s, t *be different odd primes. Put*

$$B = \mathbb{Z}[l^{-1}, p^{-1}, s^{-1}]a \oplus \mathbb{Z}[s^{-1}, t^{-1}]b \oplus \mathbb{Z}[l^{-1}, r^{-1}, t^{-1}]c$$

and

$$A = \mathbb{Z}[s^{-1}](a+b) \oplus \mathbb{Z}[l^{-1}](a+c) \oplus \mathbb{Z}[t^{-1}](b+c).$$

Then A *is a proper subgroup of* B *which is not regular in* B, *has torsion quotient* B/A *and satisfies the condition that every rank-1 summand of* A *is pure in* B.

Proof: Clearly $A \subsetneq B$ and B/A is torsion since B and A are of the same finite rank. Moreover, it is easy to check that A is completely decomposable. To prove that A is not regular in B we consider the following equation:

$$2b = (a+b) + (b+c) - (a+c) \in A$$

which shows that $\mathrm{tp}^A(2b) = \mathbb{Z} \neq \mathbb{Z}[s^{-1}, t^{-1}] = \mathrm{tp}^B(2b) = \mathrm{tp}^B(b)$ and hence A is not regular in B. Finally, easy calculations show that $\mathbb{Z}[s^{-1}](a+b)$, $\mathbb{Z}[l^{-1}](a+c)$ and $\mathbb{Z}[t^{-1}](b+c)$ are pure in B and since A is block rigid those are the only rank-1 summands of A. It follows that every rank-1 summand of A is pure in B. ∎

To state sufficient conditions for tightness we introduce the following definitions.

Definition 2.7: Let X be a torsion-free group and A a subgroup of X. Then A is called **weakly regular** in X if for all $\tau \in \mathrm{Tst}(A)$ we have $A \cap X(\tau) = A(\tau)$.

Clearly, regular subgroups are weakly regular subgroups but the converse is not true (see Example 2.12).

Definition 2.8: Let X be a torsion-free group and A a subgroup of X. Then the typeset $\mathrm{Tst}(A)$ is **tight** in $\mathrm{Tst}(X)$ if $\mathrm{Tst}(A) \subseteq M \in IT(X)$ implies $\mathrm{Tst}(A) = M$. Here,

$$IT(X) = \{\text{finite meet closure}\{\mathrm{tp}^X(x_i) : i \in I\}, x_i\text{'s maximal linearly independent}\}.$$

THEOREM 2.9: *Let X be a torsion-free group and let A be a completely decomposable subgroup of X. Then the following hold:*

(1) *If A is tight in X, then every summand of A whose typeset is linearly ordered is pure in X.*
(2) *If every rank-one summand of A is pure in X, A is weakly regular in X, $\mathrm{Tst}(A)$ is tight in $\mathrm{Tst}(A_*^X)$ and $\mathrm{Tst}(A)$ satisfies the maximum condition, then A is tight in A_*^X.*

Proof: Assume first that A is tight in X and that $A = A_1 \oplus A_2$ where A_1 has a linearly ordered typeset. Let $x \in (A_1)_*^X$, hence $nx \in A_1$ for some $n \in \mathbb{N}$. Write $A_1 = B_1 \oplus B_2$ with $nx \in B_1$ and B_1 of finite rank. Then $A' = \langle B_1, x\rangle \oplus B_2$ is completely decomposable since $\langle B_1, x\rangle$ is a finite extension of a completely decomposable group B_1 with linearly ordered typeset. Hence $A' \oplus A_2 = A$ by tightness. Thus $x \in A_1$ and it follows that $A_1 = (A_1)_*^X$.

To prove (2) assume that A is contained in a completely decomposable subgroup B of A_*^X. Without loss of generality we may assume that $B = \bigoplus_{i\in I} B_i$, where each B_i is of rank 1 and pure in X (otherwise replace B_i by its purification in X). Since the typeset of B is closed under meets we obtain that $\mathrm{Tst}(A) \subseteq \mathrm{Tst}(B)$. Note that $\mathrm{T_{cr}}(A) \subseteq \mathrm{Tst}(B)$. Thus $\mathrm{Tst}(A) = \mathrm{Tst}(B)$ since $\mathrm{Tst}(A)$ is tight in $\mathrm{Tst}(A_*^X)$. Moreover, A is weakly regular in B and hence regular in B. Finally, the quotient B/A is torsion by regularity. Thus, by Proposition 2.4, $A = B$, and hence A is tight in A_*^X. ∎

Let us remark that a tight subgroup of an almost completely decomposable group X need not have tight typeset in X as will be shown in Example 4.1. Hence there is no hope to prove the converse of Theorem 2.9 (2). However, all the examples of tight subgroups are weakly regular subgroups, so the following question remains open.

QUESTION 2.10: *If A is a tight subgroup of a torsion-free group X, is A weakly regular in X?*

COROLLARY 2.11: *Let X be a torsion-free group such that* $\mathrm{Tst}(X)$ *satisfies the maximum condition and let A be a completely decomposable subgroup of X. Then the following hold:*

(1) *If A is tight in X, then every summand of A whose typeset is linearly ordered is pure in X.*
(2) *If every rank-one summand of A is pure in X, and A is regular in X, then A is tight in A_*^X.*

Proof: All we have to show is (2) since (1) follows from Theorem 2.9(1). Thus we have to check the assumptions in Theorem 2.9(2). Since A is regular in X we obtain $\mathrm{Tst}(A) = \mathrm{Tst}(A_*^X)$ and hence $\mathrm{Tst}(A)$ is tight in $\mathrm{Tst}(A_*^X)$. Moreover, the regularity implies that also $\mathrm{Tst}(A)$ satisfies the maximum condition since $\mathrm{Tst}(A) \subseteq \mathrm{Tst}(X)$ and $\mathrm{Tst}(X)$ satisfies the maximum condition. Finally, A is trivially weakly regular in X and hence Theorem 2.9(2) implies that A is tight in A_*^X. ∎

The following example shows that tight subgroups need not be regular in general.

EXAMPLE 2.12: *Let p, q, r be different primes and let*

$$B = \mathbb{Z}[p^{-1}]a + \mathbb{Z}[q^{-1}]b + \mathbb{Z}[r^{-1}](a+b) \quad \text{and} \quad A = \mathbb{Z}[p^{-1}]a \oplus \mathbb{Z}[q^{-1}]b.$$

Then A is tight and weakly regular in B but not regular, since $\mathrm{tp}^B(a+b) = \mathbb{Z}[r^{-1}]$ *and* $\mathrm{tp}^A(a+b) = \mathbb{Z}$.

Proof: It is easy to check that every rank one summand of A is pure in B and that $\mathrm{Tst}(A)$ is tight in $\mathrm{Tst}(B)$ since

$$IT(B) = \{\{\mathbb{Z}, \mathbb{Z}[p^{-1}], \mathbb{Z}[r^{-1}]\}, \{\mathbb{Z}, \mathbb{Z}[p^{-1}], \mathbb{Z}[q^{-1}]\}, \{\mathbb{Z}, \mathbb{Z}[q^{-1}], \mathbb{Z}[r^{-1}]\}\}.$$

Moreover, A is weakly regular in B and $\mathrm{Tst}(B)$ satisfies the maximum condition. Thus Theorem 2.9 (2) implies that A is tight in B. ∎

However, we can give a characterization of the regular tight subgroups.

Definition 2.13: Let A be a completely decomposable subgroup of a torsion-free group X. The group A is called **strongly tight** in X if for every type τ, $A(\tau)$ is tight in $X(\tau)$.

The following result shows that the notions of tightness and strong tightness coincide if A is regular in X, in particular if A is a regulating subgroup of a bounded completely decomposable group X (by Lemma 3.1 a regulating subgroup of a bounded completely decomposable group is regular).

PROPOSITION 2.14: *Let X be a torsion-free group containing a completely decomposable group A such that the quotient X/A is torsion. Then the following are equivalent:*

(1) *A is tight and regular in X;*
(2) *A is strongly tight in X.*

Proof: (1)$\Longrightarrow$(2) Assume by way of contradiction that there exists a type τ such that $A(\tau)$ is not tight in $X(\tau)$. Then there exists a completely decomposable group B such that $A(\tau) \subsetneq B \subseteq X(\tau)$. Write $A = A' \oplus A(\tau), A' = \bigoplus_{\sigma \ngeq \tau} A_\tau$. Then $A' \oplus B \supsetneq A$ is completely decomposable which contradicts the tightness of A. Note that the sum $A' + B$ is direct by the regularity of A in X.

(2)$\Longrightarrow$(1) Assume that A is not tight in X. Then there exists a completely decomposable subgroup B of X such that $A \subsetneq B \subseteq X$. Hence there exists τ such that $A(\tau) \subsetneq B(\tau) \subset X(\tau)$ since otherwise we have $B(\tau) = A(\tau)$ for every type τ and thus $A = B$ which is a contradiction. Since $B(\tau)$ is completely decomposable, $A(\tau)$ is not tight in $X(\tau)$ contradicting the assumption. Therefore A is tight in X. Now let τ be a type and x an element in $X(\tau) \cap A$. Since A is strongly tight in X, the quotient $X(\tau)/A(\tau)$ is torsion. Thus $mx \in A(\tau)$ for some integer m. But $A(\tau)$ is pure in A and hence $x \in A(\tau)$. ∎

COROLLARY 2.15: *Let X be a torsion-free group and A a completely decomposable subgroup of X such that* Tst(X) *satisfies the maximum condition and X/A is torsion. Then the following are equivalent:*

(1) *A is strongly tight in X;*
(2) *A is regular in X and every rank-1 summand of A is pure in X.*

Proof: Combine Proposition 2.14 and Corollary 2.11. ∎

3. Tight subgroups of bounded completely decomposable groups

Let X be a bounded completely decomposable group. For each type τ there is a **Butler decomposition** $X(\tau) = A_\tau \oplus X^\sharp(\tau)$ where the τ-**Butler complement** A_τ is τ-homogeneous completely decomposable. The **critical typeset** $\mathrm{T_{cr}}(X)$ of X consists of all those types τ for which $X(\tau) \neq X^\sharp(\tau)$. The subgroups $\sum_{\rho \in \mathrm{T_{cr}}(X)} A_\rho$ are direct sums: $A = \bigoplus_{\rho \in \mathrm{T_{cr}}(X)} A_\rho$ and hence completely decomposable groups. Note that $\mathrm{T_{cr}}(A) = \mathrm{T_{cr}}(X)$. The subgroups A are the

regulating subgroups and Lady has shown that they are the completely decomposable subgroups of X of smallest index $\operatorname{rgi} X$ (**regulating index**) if X is of finite rank.

We show that regulating subgroups are special examples of tight subgroups provided that $\operatorname{Tst}(X)$ satisfies the maximum condition. Note that this need not be the case if there exists an ascending infinite chain in $\operatorname{Tst}(X)$ since it was shown in [MS00, Example 6.1] that in this case there are regulating subgroups A of X such that the quotient X/A is torsion-free, hence A cannot be tight in X. This indicates that without the maximum condition one can't expect much structure. Therefore, recall that the typeset of any torsion-free group of finite rank satisfies the maximum condition.

Recall that a subgroup A of a torsion-free group X is called **sharply embedded** in X if for all types τ we have $X^\sharp(\tau) \cap A = A^\sharp(\tau)$ (see [MS01]).

LEMMA 3.1: *Let X be a bounded completely decomposable group such that $\operatorname{Tst}(X)$ satisfies the maximum condition. Moreover, let A be a regulating subgroup of X and A' a subgroup of X containing A. Then A' is regular and sharply embedded in X.*

Proof: By [MS00, Lemma 4.1] it follows that the quotient X/A is torsion. Moreover, by [MS00, Proposition 3.1.2] we have $X(\tau) \cap A = A(\tau)$ for every type $\tau \in \operatorname{Tst}(X)$. Now let τ be any type and $x \in A' \cap X(\tau)$. Then there exists an integer m such that $mx \in A$ since X/A is torsion. Hence $mx \in A \cap X(\sigma) = A(\sigma)$, where $\sigma = \operatorname{tp}^X(x) \geq \tau$. But $A(\sigma) \subseteq A(\tau) \subseteq A'(\tau)$ is pure in A', consequently, $x \in A'(\tau)$. Thus A' is regular in X.

It remains to prove that A' is also sharply embedded in X. Let τ be a type and $x \in A' \cap X^\sharp(\tau)$. Then there exists an integer m such that

$$mx \in A \cap X^\sharp(\tau) \subseteq A \cap X(\tau) = A(\tau)$$

by the above. Write $A(\tau) = A_\tau \oplus A^\sharp(\tau)$ with A_τ τ-homogeneous completely decomposable. Then $mx = a_\tau + a_\tau^\sharp$ for some $a_\tau \in A_\tau$ and $a_\tau^\sharp \in A^\sharp(\tau)$. Hence $a_\tau = mx - a_\tau^\sharp \in X^\sharp(\tau) \cap A_\tau = \{0\}$ which implies that $mx = a_\tau^\sharp \in A^\sharp(\tau) \subseteq A'^\sharp(\tau)$. Consequently $x \in A'^\sharp(\tau)$ since $A'^\sharp(\tau)$ is pure in A'. ∎

We are now able to prove that every regulating subgroup in a bounded completely decomposable group whose typeset satisfies the maximum condition is tight (even strongly tight). This and the final Lemma 3.6 of this section answer partially a question posed by Mader and the second author in [MS00, Question 2].

PROPOSITION 3.2: *Let X be a bounded completely decomposable group whose typeset satisfies the maximum condition. Then every regulating subgroup of X is strongly tight in X.*

Proof: Let A be a regulating subgroup of X and A' be a completely decomposable subgroup of X which contains A. We will show that for every $\tau \in \mathrm{T}_{\mathrm{cr}}(A')$, $A'(\tau) = A_\tau \oplus A'^\sharp(\tau)$. Let $x \in A'(\tau) \subseteq X(\tau)$. Then there exist $a_\tau \in A_\tau$ and $x_\tau^\sharp \in X^\sharp(\tau)$ such that $x = a_\tau + x_\tau^\sharp$ and therefore $x_\tau^\sharp = x - a_\tau \in A'(\tau) \cap X^\sharp(\tau) = A'^\sharp(\tau)$ by Lemma 3.1. Thus $x \in A_\tau + A'^\sharp(\tau)$. To show that the sum $A_\tau + A'^\sharp$ is also direct let $x \in A_\tau \cap A'^\sharp(\tau)$. Then $x \in A_\tau \cap X^\sharp(\tau) = 0$. Hence by [MMR94, Lemma 2.3] $A' = A$ which means that A is tight in X. Since A is regular in X by Lemma 3.1 it follows that A is strongly tight in X by Proposition 2.14. ∎

We now generalize literally some results on almost completely decomposable groups from [D01a] and [D01b] to the class of bounded completely decomposable groups.

LEMMA 3.3: *Let X be a bounded completely decomposable group such that $\mathrm{Tst}(X)$ satisfies the maximum condition and let A be a completely decomposable subgroup of bounded index in X. Then A is not tight in X if and only if there exists a critical type $\tau \in \mathrm{Tst}(X)$ and an element $x \in X(\tau)\backslash(A_\tau \oplus X^\sharp(\tau))$ of prime order modulo A.*

Proof: Assume first that A is not tight in X. Then A is not regulating by Proposition 3.2 and hence $X(\tau) \neq A_\tau \oplus X^\sharp(\tau)$ for some critical type type $\tau \in \mathrm{Tst}(X)$. Clearly there exists an element $x \in X(\tau)\backslash(A_\tau \oplus X^\sharp(\tau))$ which is of prime order.

Conversely, assume that there exists a critical type $\tau \in \mathrm{Tst}(X)$ and an element $x \in X(\tau)\backslash(A_\tau \oplus X^\sharp(\tau))$ which is of prime order p modulo A. We will show that $B = \langle A, x\rangle$ is completely decomposable which contradicts the tightness of A. Let $x = \frac{1}{p}(a_\tau + a^\sharp)$ where $a_\tau \in A_\tau$ and $a^\sharp \in A^\sharp(\tau)$. Thus $\mathrm{ht}_p^A(a_\tau) = 0$ and since $\mathrm{tp}^A(a_\tau) \leq \mathrm{tp}^A(a^\sharp)$ there exists a natural number k such that $\chi^A(a_\tau) \leq \chi^A(ka^\sharp)$ and $\gcd(k,p) = 1$. Thus there exist natural numbers s and r such that $rp+sk = 1$. Since $ra^\sharp \in A$ we obtain $B = \langle A, x - rx\rangle$. Write

$$x - ra^\sharp = \frac{1}{p}(a_\tau + (1 - rp)a^\sharp) = \frac{1}{p}(a_\tau + ska^\sharp).$$

As $\chi^A(a_\tau) \leq \chi^A(ska^\sharp)$ there exists a homomorphism $\phi \in \mathrm{Hom}(A_\tau, A^\sharp(\tau))$ such that $\phi(a_\tau) = ska^\sharp$. It follows that $B = (A_\tau(1 + \phi))_*^A \oplus \bigoplus_{\sigma \neq \tau} A_\sigma$ and since

$A_\tau(1+\phi) \cong A_\tau$ we obtain that $(A_\tau(1+\phi))_*^A$ is completely decomposable as a bounded extension of a homogeneous completely decomposable group. Therefore also B is completely decomposable. ∎

COROLLARY 3.4: *Let X be a bounded completely decomposable group such that* $\mathrm{Tst}(X)$ *satisfies the maximum condition containing a tight subgroup A such that $k(X/A) = 0$ for some square-free integer k. Then A is regulating in X.*

Proof: Assume that A is not regulating in X and let τ be a critical type of X such that $X(\tau) \neq A_\tau \oplus X^\sharp(\tau)$. Let $x \in X(\tau)\setminus(A_\tau \oplus X^\sharp(\tau))$ be of order m. Then m must be square-free and moreover, $\gcd(m/p_1, \ldots, m/p_k) = 1$, where $p_1, \ldots, p_k$ are the prime divisors of m. Hence $m = \sum_{i=1}^k r_i(m/p_i)$ for suitable integers r_i. Define $x_i = x(m/p_i)$ and note that x_i has order p_i modulo A. Since A is tight in X we know by Lemma 3.3 that $x_i \in A_\tau \oplus X^\sharp(\tau)$ for all i. Thus $x = \sum_{i=1}^k r_i x_i \in A_\tau \oplus X^\sharp(\tau)$ — a contradiction. ∎

COROLLARY 3.5: *Let X be a bounded completely decomposable group such that* $\mathrm{Tst}(X)$ *satisfies the maximum condition containing a completely decomposable subgroup A such that $k(X/A) = 0$ for some square-free integer k. Then A is contained in a regulating subgroup of X.*

Proof: By Corollary 3.4 it is sufficient to prove that A is contained in a tight subgroup of X. Let

$$\mathfrak{M} = \{D \subseteq X : D \text{ completely decomposable}, A \subseteq D\}.$$

Then $\mathfrak{M}$ is a partially ordered set by inclusion containing A. Let $\langle D_i : i \in I\rangle$ be an ascending chain in $\mathfrak{M}$ and put $D = \bigcup_{i\in I} D_i$, so that $A \subseteq D \subseteq X$. We will show that D is torsionless, i.e., for all types τ we have $D^\sharp(\tau) = D^*(\tau)$. Let τ be a type and $x \in D$ such that $mx \in D^*(\tau)$ for some natural number m. Note that A is sharply embedded into X, hence we have

$$D^*(\tau) \cap D_i = D_i^*(\tau) = D_i^\sharp(\tau)$$

for every $i \in I$. By definition of D there exists $i \in I$ such that $x \in D_i$. Thus $mx \in D_i \cap D^*(\tau) = D_i^*(\tau)$. But $D_i^*(\tau)$ is pure in D_i, hence $x \in D_i^*(\tau) \subseteq D^*(\tau)$ and therefore D is torsionless. Since D is a bounded completely decomposable group as it contains A we conclude from [MS00, Proposition 4.3] that D is completely decomposable. By Zorn's lemma $\mathfrak{M}$ contains maximal elements and hence A can be embedded into a tight subgroup of X ∎

In the proof of Corollary 3.5 it turns out that any completely decomposable subgroup of bounded quotient in a bounded completely decomposable group X can be embedded into a tight subgroup of X provided that $\mathrm{Tst}(X)$ satisfies the maximum condition. This is no longer the case for arbitrary bounded completely decomposable groups. In fact, if $\mathrm{Tst}(X)$ is an ascending chain, then this can only happen in the completely decomposable case as the next lemma shows.

LEMMA 3.6: *Let X be a bounded completely decomposable group such that $\mathrm{Tst}(X)$ is linearly ordered. Then X is completely decomposable if and only if it contains a tight subgroup.*

Proof: Clearly, if X is completely decomposable, then X is tight in itself. Thus assume that X contains a tight subgroup A. Note that $\mathrm{Tst}(A) \subseteq \mathrm{Tst}(X)$ and hence linearly ordered. By Theorem 2.9 A is pure in X and hence $X = A$ which is completely decomposable. ∎

We conclude this section with the definition of sharply tight subgroups.

Definition 3.7: Let X be a torsion-free group and A a completely decomposable subgroup of X. Then A is called **sharply tight** in X if A is tight in X and for all types τ also $A^\sharp(\tau)$ is tight in $X^\sharp(\tau)$.

Remark 3.8: It is easy to see that for all types τ, $A^\sharp(\tau)$ is tight in $X^\sharp(\tau)$ and $\mathbb{Z} \notin \mathrm{T_{cr}}(A)$ implies that A is sharply tight in X but the converse is not known.

We have a first proposition which is the analogue to Proposition 2.14.

PROPOSITION 3.9: *Let X be a torsion-free group and A a completely decomposable subgroup of X such that X/A is torsion. Then the following are equivalent:*

(1) *A is sharply tight in X;*
(2) *A is sharply embedded in X and tight in X.*

Proof: (2)$\Longrightarrow$(1) Assume by way of contradiction that there exists a type τ such that $A^\sharp(\tau)$ is not tight in $X^\sharp(\tau)$. Then there exists a completely decomposable subgroup B such that $A^\sharp(\tau) \subsetneq B \subseteq X^\sharp(\tau)$. Write $A = A' \oplus A^\sharp(\tau)$, $A' = \bigoplus_{\sigma \not> \tau} A_\tau$. Then $A' \oplus B \supsetneq A$ which contradicts the tightness of A. Note that the sum $A' + B$ is direct since A is sharply embedded in X.

(1)$\Longrightarrow$(2) By definition A is tight in X. Assume that A is not sharply embedded in X, and let τ be a type such that $A^\sharp(\tau) \subsetneq X^\sharp(\tau) \cap A$. Choose an element x in $X^\sharp(\tau) \cap A$ but not in $A^\sharp(\tau)$ and write $x = \bar{a}_\tau + a_\tau$ where $\bar{a}_\tau \in \bigoplus_{\sigma \not> \tau} A_\tau$ and $a_\tau \in A^\sharp(\tau)$. Then $x - a_\tau = \bar{a}_\tau \in X^\sharp(\tau) \cap A$ but not in $A^\sharp(\tau)$. Therefore $A^\sharp(\tau)$ is strictly contained in the completely decomposable subgroup $\langle \bar{a}_\tau \rangle \oplus A^\sharp(\tau)$ which is impossible as $A^\sharp(\tau)$ is tight in $X^\sharp(\tau)$. ∎

LEMMA 3.10: *Let X be a bounded completely decomposable group such that $\mathrm{Tst}(X)$ satisfies the maximum condition and let A be a regulating subgroup of X. Then A is sharply tight in X.*

Proof: By Proposition 3.2 we know that every regulating subgroup is tight. Moreover, by Lemma 3.1, A is sharply embedded in X and hence Proposition 3.9 shows that A is sharply tight in X. ∎

4. Tight subgroups of finite index in almost completely decomposable groups

In this section we characterize tight subgroups of almost completely decomposable groups which have finite index. Recall that a subgroup A of a torsion-free group X is called **critically regular in** X if $X^\sharp(\tau) \cap A(\tau) = A^\sharp(\tau)$ for all types τ. Moreover, it is called **strongly regular in** X if it is regular and critically regular in X (see [MM90]).

In [BMO00] the authors claimed that every tight subgroup of an almost completely decomposable group X has finite index in X but the proof is not correct, since they assumed that any such tight subgroup is strongly regular in X which is not justified. We first give a counterexample which is a courtesy of Prof. C. Vinsonhaler. However, as mentioned in the introduction all the results in [BMO00] remain true if one restricts to tight subgroups of finite index.

EXAMPLE 4.1: *Let p be a prime and τ_1, τ_2, τ_3 incomparable types (rational groups of incomparable type) such that $1/p \notin \tau_1, \tau_2, \tau_3$. Put $\tau_{i,j} = \tau_i \cap \tau_j$ for $i, j = 1, \ldots, 3$ and assume that $\tau_{1,2}, \tau_{1,3}$ and $\tau_{2,3}$ are incomparable. Moreover, let $\tau_{1,2} \cap \tau_{1,3} = \tau_{1,2} \cap \tau_{2,3} = \tau_{1,3} \cap \tau_{2,3} = \mathbb{Z}$. Put*

$$X = \tau_1 x_1 \oplus \tau_2 x_2 \oplus \tau_3 x_3 + \frac{1}{p}\tau_{1,2}(x_1 + x_2) + \frac{1}{p}\tau_{2,3}(x_2 + x_3)$$

and

$$A = \frac{1}{p}\tau_{1,2}(x_1 + x_2) \oplus \frac{1}{p}\tau_{2,3}(x_2 + x_3) \oplus \tau_{1,3}(x_1 + x_3).$$

Then X is an almost completely decomposable group, A is tight in X but X/A is unbounded.

Proof: First note that X/A is unbounded and that A is in fact a direct sum. Moreover, A is weakly regular in X. Since $\frac{1}{p}\tau_{1,2}(x_1 + x_2)$, $\frac{1}{p}\tau_{2,3}(x_2 + x_3)$ and

$\tau_{1,3}(x_1 + x_3)$ are pure in X and A is block rigid it follows that every rank-1 summand of A is pure in X. Unfortunately, $\mathrm{Tst}(A) = \{\tau_{1,2}, \tau_{1,3}, \tau_{2,3}, \mathbb{Z}\}$ is not tight in $\mathrm{Tst}(X)$, hence we cannot apply our Theorem 2.9 directly. Thus we have to be more careful. Assume that $A \subsetneq B \subseteq X$, where B is completely decomposable. Moreover, assume that $\mathrm{Tst}(A) \subsetneq \mathrm{Tst}(B) \subsetneq \mathrm{Tst}(X)$. As an example we discuss the case $\mathrm{Tst}(B) = \mathrm{Tst}(A) \cup \{\tau_1\}$. Then $\mathrm{T_{cr}}(B) = \{\tau_1, \tau_{1,2}, \tau_{2,3}\}$ or $\mathrm{T_{cr}}(B) = \{\tau_1, \tau_{1,3}, \tau_{2,3}\}$ which contradicts $\{\tau_{1,2}, \tau_{1,3}\} \subseteq \mathrm{Tst}(A) \subseteq \mathrm{Tst}(B)$. Thus $\mathrm{Tst}(A) = \mathrm{Tst}(B)$ or $\mathrm{Tst}(B) = \mathrm{Tst}(X)$. In the first case the proof of Theorem 2.9 gives $A = B$. In the latter case we have $\mathrm{T_{cr}}(B) = \mathrm{T_{cr}}(X) = \{\tau_1, \tau_2, \tau_3\}$ and hence $B \subseteq \tau_1 x_1 \oplus \tau_2 x_2 \oplus \tau_3 x_3$. Therefore, $A \subseteq B \subseteq \tau_1 x_1 \oplus \tau_2 x_2 \oplus \tau_3 x_3$ — a contradiction. Consequently, A is tight in X. ∎

We now state a corrected version of [BMO00, Lemma 2.1] and recall the proof to point out where strong regularity is needed.

LEMMA 4.2: *Let X be an almost completely decomposable group and A a tight subgroup of X which is strongly regular in X. Then X/A is finite.*

Proof: We use induction on rank. If the rank of X is one then clearly $X = A$ and hence X/A is finite. Let X be an almost completely decomposable group and assume that the claim holds for almost completely decomposable groups of rank less than $\mathrm{rk}\, X$. Let A be a tight subgroup of X. Choose a minimal critical type $\mu \in \mathrm{Tst}(X)$ and observe that the index $[X : (X(\mu) + X[\mu])]$ is finite since μ is minimal. Therefore it suffices to show that $[X(\mu) : A(\mu)]$ and $[X[\mu] : A[\mu]]$ are both finite. By Proposition 2.14 we know that $A(\mu)$ is tight in $X(\mu)$. We will show that also $A[\mu]$ is tight in $X[\mu]$. Assume that there exists a completely decomposable group $A[\mu] \subsetneq B \subseteq X[\mu]$. Since μ is minimal in $\mathrm{Tst}(A)$ we have $A = A_\mu \oplus A[\mu]$. We claim that $A_\mu + B$ is a direct sum $A_\mu \oplus B$ which would contradict the tightness of A in X. Let $x \in B \cap A_\mu$. Then $x \in X(\mu) \cap X[\mu] = X^\sharp(\mu)$ (see [MS00, Lemma 2.3]). Thus $x \in A \cap X^\sharp(\mu) = A^\sharp(\mu)$. Hence $x \in A_\mu \cap A^\sharp(\mu) = 0$. Thus $A[\mu]$ is tight in $X[\mu]$. Therefore $[X(\mu) : A(\mu)]$ and $[X[\mu] : A[\mu]]$ are both finite by induction unless $X = X(\mu)$ or $X = X[\mu]$. The latter case cannot occur since $X[\mu] = (\bigoplus_{\rho \neq \mu} A_\rho)_*^X$ as $A[\mu]$ is tight in $X[\mu]$, μ being minimal in $\mathrm{T_{cr}}(X)$, and $A_\mu \neq 0$. In the former case, $X = X_\mu \oplus X^\sharp(\mu)$ and $A = A_\mu \oplus A^\sharp(\mu)$ since obviously $\mathrm{T_{cr}}(A) \subset \mathrm{Tst}(X)$. Certainly X/A is a torsion group and it follows from

$$\frac{X}{A} = \frac{X_\mu \oplus X^\sharp(\mu)}{A_\mu \oplus A^\sharp(\mu)}$$

by an easy rank calculation that $\mathrm{rk}\, A_\mu = \mathrm{rk}\, X_\mu$ and hence $X_\mu \cong A_\mu$. Now consider the short exact sequence

$$\frac{A+X^\sharp(\mu)}{A} \rightarrowtail \frac{X}{A} \twoheadrightarrow \frac{X+X^\sharp(\mu)}{A+X^\sharp(\mu)}.$$

The left end is isomorphic to

$$\frac{X^\sharp(\mu)}{A\cap X^\sharp(\mu)} = \frac{X^\sharp(\mu)}{A^\sharp(\mu)}$$

and finite by induction. To show that the right end of the short exact sequence is also finite, and so finishing the proof, set $Y = A_\mu \oplus X^\sharp(\mu)$. Then

$$\frac{X+X^\sharp(\mu)}{A+X^\sharp(\mu)} = \frac{X_\mu \oplus X^\sharp(\mu)}{A_\mu \oplus X^\sharp(\mu)} = \frac{X_\mu \oplus X^\sharp(\mu)}{(Y\cap X_\mu)\oplus X^\sharp(\mu)} \cong \frac{X_\mu}{X_\mu \cap Y}.$$

Since $Y = A_\mu \oplus X^\sharp(\mu) = (X_\mu \cap Y) \oplus X^\sharp(\mu)$, it follows that $X_\mu \cap Y \cong A_\mu \cong X_\mu$ and therefore ([Mad99, Proposition 2.1.3]) the right hand end of the short exact sequence is also finite. ∎

LEMMA 4.3: *Let X be an almost completely decomposable group and A a tight subgroup of X. Then the following are equivalent:*

(1) *A has finite index in X;*

(2) *A is sharply tight in X.*

Proof: If A is tight in X and has finite index, then clearly A is sharply embedded in X. Hence Proposition 3.9 shows that A is sharply tight in X. Conversely, if A is sharply tight in X, then A is tight in X and A is sharply embedded in X. By [MS01, Corollary 4.24] it follows that A is strongly regular in X and thus Lemma 4.2 proves that A has finite index in X. ∎

Observe that Lemma 3.10 shows that the above Lemma 4.3 does not hold anymore if we replace almost completely decomposable by bounded completely decomposable. Note that there are regulating subgroups of a bounded completely decomposable group X which have unbounded quotient in X (see [MS00, Example 5.2]).

Note also that in contrast to regulating subgroups not every tight subgroup in an almost completely decomposable group has to be sharply embedded as Example 4.1 shows.

To conclude this section we give another characterization of the tight subgroups in almost completely decomposable groups which are of finite index in the special case when the critical typeset is an antichain.

PROPOSITION 4.4: *Let X be an almost completely decomposable group with critical typeset an antichain and A a tight subgroup of X. Then the following are equivalent:*

(1) *A is of finite index in X;*
(2) *for all $\tau \in \mathrm{T_{cr}}(X)$ we have $X(\tau) = A_\tau$;*
(3) *for all $\tau \in \mathrm{T_{cr}}(X)$ we have $\mathrm{rk}(A_\tau) = \mathrm{rk}(X(\tau))$.*

Proof: Clearly, if A is of finite index in X, then $\mathrm{T_{cr}}(A) = \mathrm{T_{cr}}(X)$ by regularity. Note that a type $\tau \in \mathrm{Tst}(X)$ is critical if and only if it is maximal. Moreover, for $\tau \in \mathrm{T_{cr}}(X)$ we have $X(\tau)/A(\tau) = X(\tau)/A_\tau$, which is torsion by regularity and at the same time torsion-free by Corollary 2.11 since A is tight in X. Thus $A_\tau = X(\tau)$ and (2) and (3) are satisfied. Moreover, (2) implies (3). Therefore assume that (3) holds. Let $\tau \in \mathrm{T_{cr}}(X) \subseteq \mathrm{T_{cr}}(A)$; then τ is maximal in $\mathrm{Tst}(A)$ since $\mathrm{T_{cr}}(X)$ is an antichain. Hence $A(\tau) = A_\tau \lhd X(\tau)$ which is homogeneous completely decomposable of type τ. Thus A_τ is a summand of $X(\tau)$ and so $A_\tau = X(\tau)$ since they have the same ranks. We now prove that A is sharply embedded in X and thus it is sharply tight in X by Proposition 3.9, which shows that X/A is finite by Corollary 4.3. Let τ be any type and $x \in A \cap X^\sharp(\tau)$. We induct on the depth of $\sigma = \mathrm{tp}^X(x)$. If σ is maximal (hence critical) in $\mathrm{Tst}(X)$, then $\sigma \in \mathrm{Tst}(A)$ and, moreover, $\sigma \in \mathrm{T_{cr}}(A)$ by assumption. If $\sigma = \tau$, then $X^\sharp(\tau) = 0 = A^\sharp(\tau)$ and there is nothing to show. Thus assume $\sigma > \tau$. By the discussion above it follows that $x \in X(\sigma) = A_\sigma \subseteq A^\sharp(\tau)$. Now let $\sigma = \mathrm{tp}^X(x)$ be not maximal and assume that for all $y \in A \cap X^\sharp(\tau)$ with $\mathrm{tp}^X(y) > \sigma$ we have $y \in A^\sharp(\tau)$. Since $\mathrm{T_{cr}}(X)$ is an antichain, σ cannot be critical in $\mathrm{Tst}(X)$. Thus $x \in X^\sharp(\sigma)$. Write $nx = \sum_{\rho>\sigma} y_\rho$ for some integer n, where $y_\rho \in X(\rho)$. Since the quotient X/A is torsion there is a natural number m such that for all $\rho > \sigma$ we have $my_\rho \in A$. Thus $my_\rho \in A \cap X^\sharp(\tau)$ and therefore $my_\rho \in A^\sharp(\tau)$ by induction hypothesis. Thus $mnx \in A^\sharp(\tau)$ which is pure in A and therefore also $x \in A^\sharp(\tau)$. ∎

We have some open questions:

QUESTION 4.5:

(1) *What is the intersection of all tight subgroups of various torsion-free groups?*
(2) *Which completely decomposable subgroups of an almost or bounded completely decomposable group can be embedded into a tight subgroup?*
(3) *Which bounded completely decomposable groups whose typeset does not satisfy the maximum condition have tight subgroups?*

References

[Arn81] D. M. Arnold, *Pure Subgroups of Finite Rank Completely Decomposable Groups*, Lecture Notes in Mathematics **874**, Springer-Verlag, Berlin, 1981, pp. 1–31.

[Arn82] D. M. Arnold, *Finite Rank Torsion Free Abelian Groups and Rings*, Lecture Notes in Mathematics **931**, Springer-Verlag, Berlin, 1982.

[BMO00] K. Benabdallah, A. Mader and M. A. Ould-Beddi, *Tight subgroups in almost completely decomposable groups*, Journal of Algebra **225** (2000), 501–516.

[Bur84] R. Burkhardt, *On a special class of almost completely decomposable groups I*, in *Abelian Groups and Modules, Proceedings of the Udine Conference 1984*, CISM Courses and Lecture Notes, Volume 287, Springer-Verlag, Berlin, 1984, pp. 141–150.

[D01a] U. Dittmann, *Coset Types and Tight Subgroups of Almost Completely Decomposable Groups*, PhD thesis, Julius Maximilian Universität Würzburg, 2001.

[D01b] U. Dittmann, *Tight subgroups of almost completely decomposable groups*, in *Proceedings of Second Honolulu Conference on Abelian Groups and Modules*, 2001, to appear.

[Du01] M. Dugas, *BCD-groups with typeset* (1, 2), Forum Mathematica **13** (2001), 143–148.

[El96] A. Elter, *Torsionsfreie Gruppen mit linear geordneter Typenmenge*, PhD thesis, Universität Essen, 1996.

[Fuc73] L. Fuchs, *Infinite Abelian Groups, Vol. I, II*, Academic Press, New York, 1970 and 1973.

[Lad74] E. L. Lady, *Almost completely decomposable torsion-free abelian groups*, Proceedings of the American Mathematical Society **45** (1974), 41–47.

[Mad92] A. Mader, *Almost Completely Decomposable Abelian Groups*, Photocopied notes, Université de Montréal, 1992.

[Mad95] A. Mader, *Almost completely decomposable torsion-free abelian groups*, in *Abelian Groups and Modules, Proceedings of the 1994 Padova Conference*, Kluwer Academic Publishers, Dordrecht, 1995, pp. 343–366.

[Mad99] A. Mader, *Almost Completely Decomposable Groups, Algebra, Logic and Applications*, Volume 13, Gordon and Breach Science Publishers, London, 1999.

[MMR94] A. Mader, O. Mutzbauer and K. M. Rangaswamy, *A generalization of Butler groups*, Contemporary Mathematics **171** (1994), 257–275.

[MS00] A. Mader and L. Strüngmann, *Bounded essential extensions of completely decomposable groups*, Journal of Algebra **229** (2000), 205–233.

[MS01] A. Mader and L. Strüngmann, *scd-subgroups of completely decomposable groups*, submitted, 2001.

[MV95] A. Mader and C. Vinsonhaler, *Almost completely decomposable groups with a cyclic regulating quotient*, Journal of Algebra **177** (1995), 463–492.

[MM90] E. Müller and O. Mutzbauer, *Regularität in torsionsfreien abelschen Gruppen*, Czechoslovak Mathematical Journal **42** (1990), 279–288.